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The Effect of Temperature on the Deformation of Infinite or Semi-Infinite Elastic Body. II.

By

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The Effect of Temperature on the Deformation of Infinite or Semi-Infinite Elastic Body, II.[(1)]

By

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After publishing my first paper, I intended to discuss a corresponding problem in geophysics. Here I have treated the effect of the daily and annual changes of temperature in the earth crust. I have concluded that:

(I) There are also the daily and annual changes of the deformation of the earth crust.

(II) The magnitudes of deformation at the surface are 10^{-3} cm for a day and 10^{-1} cm for a year, at most.

(III) The magnitude of deformation decreases exponentially as the depth increases.

(IV) The phase difference between the corresponding harmonics of temperature and deformation is $\frac{\pi}{4}$ i.e., 3 hours for the daily change and 1,5 months for the annual change.

In general, thermal stresses do not exist in the earth when the distribution of *invariable* temperature is *any* given function of depth only. Only the normal stress of small magnitude exists in the earth when the earth temperature is a function of the depth as well as the *time*.

Since the effect of temperature on the deformation of an elastic body is mainly determined by its distribution at the instant under consideration; the most general solution which is described in this paper is almost identical with that for the deformation of elastic body for any given distribution of invariable temperature. The solutions are constructed so as to satisfy the condition that the surface is free from traction.

§1 If we start from the state of invariable temperature with any given invariable distribution of heat $f(x)$ in a semi-elastic body, then the distribution[2)] of temperature, displacement and normal stress are given by,

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(1) The part I of the present paper was published in Vol. IV of this magazine.

(2) Geophys. Mag., Vol. IV, p. 299 (1931).

$$T=\frac{2}{\pi}\int_0^\infty f(\xi)d\xi\int_0^\infty e^{-a^2\alpha^2t}\sin\alpha x\sin\alpha\xi\,d\alpha, \tag{1.1}$$

$$u=\frac{2}{\pi}\int_0^\infty f(\xi)d\xi\int_0^\infty\frac{-\frac{\lambda+2\mu}{3}c}{\rho a^4\alpha^3+(\lambda+2\mu)\alpha}e^{-a^2\alpha^2t}\cos\alpha x\sin\alpha\xi\,d\alpha \tag{1.2}$$

and

$$X_x=(\lambda+2\mu)\frac{2}{\pi}\int_0^\infty f(\xi)d\xi\int_0^\infty\frac{\frac{\lambda+2\mu}{3}c}{\rho a^4\alpha^2+(\lambda+2\mu)}e^{-a^2\alpha^2t}\sin\alpha x\sin\alpha\xi\,d\alpha$$
$$-\frac{2\frac{\lambda+2\mu}{3}c}{\pi}\int_0^\infty f(\xi)d\xi\int_0^\infty e^{-a^2\alpha^2t}\sin\alpha x\sin\alpha\xi\,d\alpha, \tag{1.3}$$

which satisfies the boundary condition that the surface $x=0$ is free from traction.

Since Lamé's constants λ and μ are very large compared with the density ρ and the diffusivity a^2, we may approximately put

$$u=-\frac{2c}{3\pi}\int_0^\infty f(\xi)d\xi\int_0^\infty\frac{1}{\alpha}e^{-a^2\alpha^2t}\cos\alpha x\sin\alpha\xi\,d\alpha,$$

$$X_x=0,$$

which can be obtained by finding the solution for the displacement corresponds to the distribution of invariable temperature as

$$T=\frac{2}{\pi}\int_0^\infty f(\xi)d\xi\int_0^\infty e^{-a^2\alpha^2t}\sin\alpha x\sin\alpha\xi\,d\alpha,$$

t being considered as a parameter. As described above, the effects of variable temperature on the deformation and stress of an elastic body are mainly determined by the distribution of temperature at the instant under consideration[1], we may say that if the deformations and stresses for any given distribution of invariable temperature are obtained, then the problem is solved almost perfectly.

In general if we put $\rho=t=0$ in (1.1), (1.2) and (1.3), we get as

$$T=\frac{2}{\pi}\int_0^\infty f(\xi)d\xi\int_0^\infty\sin\alpha x\sin\alpha\xi\,d\alpha, \tag{1.4}$$

$$u=-\frac{2c}{3\pi}\int_0^\infty f(\xi)d\xi\int_0^\infty\frac{1}{\alpha}\cos\alpha x\sin\alpha\xi\,d\alpha, \tag{1.5}$$

and

$$X_x=0. \tag{1.6}$$

(1) Prof. K. Sezawa insisted this opinion before the January Meeting of the Earthquake Research Institue, Tôkyô. I owed also to Mr. G. Nishimura's discussions read before that Meetings.

(1.5) and (1.6) express the displacement and normal stress corresponding any given distribution of heat.

§2. If the temperature at the surface of earth is a periodic function whose period is one day or one year, it can be expressed by a Fourier's series of the form

$$F(t)=b_0+\sum_{m=1}^{\infty} b_m \sin(m\omega t+\kappa_m),$$

where the axis of x is taken vertically downwards, b_0 is the mean temperature, ω stands for $\dfrac{2\pi}{24\times60\times60}$ or $\dfrac{2\pi}{365\times24\times60\times60}\,\dfrac{1}{\sec}$, and b_0 and b_m are the constants properly determined from the observations.

Then the corresponding distribution of temperature is given by

$$T=b_0+\sum_{m=1}^{\infty} b_m e^{-\frac{x}{a}\sqrt{\frac{m\omega}{2}}} \sin\left(m\omega t-\frac{x}{a}\sqrt{\frac{m\omega}{2}}+\kappa_m\right).^{(1)}$$

As shown above, the effect due to the time-variation of temperature is very small and may be neglected, hence we may safely discuss the daily or annual periodic deformation of the earth by calculating the deformation corresponds to the distribution of temperature for each instant during the periodic process. Thus we may put in (1.5) as

$$f(\xi)=b_0+\sum_{m=1}^{\infty} b_m e^{-\frac{\xi}{a}\sqrt{\frac{m\omega}{2}}} \sin\left(m\omega t-\frac{\xi}{a}\sqrt{\frac{m\omega}{2}}+\kappa_m\right),$$

and may suppose t as a parameter. Thus we have, even if $f(x)$ is not zero at $x=0$,

$$u=-\frac{2c}{3\pi}\int_0^{\infty}\left\{b_0+\sum_{m=1}^{\infty} b_m e^{-\frac{\xi}{a}\sqrt{\frac{m\omega}{2}}} \sin\left(m\omega t-\frac{\xi}{a}\sqrt{\frac{m\omega}{2}}+\kappa_m\right)\right\}d\xi\int_0^{\infty}\frac{1}{\alpha}\cos\alpha x\sin\alpha\xi\,d\alpha, \tag{2.1}$$

$$X_x=0. \tag{2.2}$$

Using the relations:

$$\int_0^{\infty}\frac{\sin p\alpha\cos q\alpha}{\alpha}d\alpha=0 \quad \text{if} \quad q<-p \quad \text{or} \quad q>p,$$

$$\frac{\pi}{4} \quad \text{if} \quad q=-p \quad \text{or} \quad q=p,$$

$$\frac{\pi}{2} \quad \text{if} \quad -p<q<p,$$

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(1) W. E. Byerly; Fourier's Series etc., p. 89 (1893).

and
$$\int e^{px}\sin qx\,dx=\frac{e^{px}(p\sin qx-q\cos qx)}{p^2+q^2},$$

we have

$$u=-\frac{c}{3}b_0[\xi]_x^\infty-\frac{ac}{3}\sum_{m=1}^{\infty}b_m\frac{e^{-\frac{x}{a}\sqrt{\frac{m\omega}{2}}}}{\sqrt{m\omega}}\sin\left(m\omega t-\frac{x}{a}\sqrt{\frac{m\omega}{2}}+\kappa_m-\frac{\pi}{4}\right).\quad(2.3)$$

The above expression contains the term corresponding to the infinity. To avoid the difficulty we must take the state corresponding to the uniform temperature b_0 as the standard states, i.e., we must subtract the term for deformation corresponding for the constant temperature b_0, which is given by

$$-\frac{c}{3}b_0[\xi]_x^\infty.$$

Thus we have

$$u=-\frac{ac}{3}\sum_{m=1}^{\infty}b_m\frac{e^{-\frac{x}{a}\sqrt{\frac{m\omega}{2}}}}{\sqrt{m\omega}}\sin\left(m\omega t-\frac{x}{a}\sqrt{\frac{m\omega}{2}}+\kappa_m-\frac{\pi}{4}\right).\quad(2.4)$$

We can easily check (2.3) and (2.4) satisfies the thermoelastic equation

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}=\frac{\lambda+2\mu}{3}c\frac{\partial T}{\partial x}.$$

Thus we may conclude as follows :—

(I) The displacement due to the earth temperature changes periodically with the period, one day or one year.

(II) The magnitude of displacement decreases exponentially with increasing depth.

(III) The phase difference between the corresponding harmonics of temperature and displacement is $\frac{\pi}{4}$, or 3 hours for daily change and 1,5 months for annual change. This fact is not inconsistent with the opinion that the effect of temperature on the deformation is mainly determined by the distribution of temperature at the instant under consideration. This is the result of the elongation and contraction of masses in the deeper interior of the earth.

(IV) For any depth, the amplitude of the earth temperature wave can be reduced to that of the deformation by multiplying

$$\frac{ac}{3}\frac{1}{\sqrt{m\omega}}\quad\left(\text{dimension}:\ \frac{\text{length}}{\text{degree}}\right).$$

This is the results of the elongation and contraction of the masses in the deeper layers of the earth.

(1) $(-u)$ express the displacement with regard to the x-axis taken vertically upwards.

Since

$$a^2 \fallingdotseq 0.01$$

$$\frac{c}{3} \fallingdotseq 0.1 \times 10^{-4}$$

$$\omega = \begin{cases} \dfrac{2\pi}{24 \times 60 \times 60} & \text{for daily change} \\ \dfrac{2\pi}{365 \times 24 \times 60 \times 60} & \text{for annual change} \end{cases}$$

$$b_1 \fallingdotseq \begin{cases} 10°\text{C} & \text{for daily change} \\ 30°\text{C} & \text{for annual change} \end{cases}$$

approximately, the daily displacement at the surface is 10^{-3} cm and the annual deformation is 10^{-1} cm, at most.

(V) The thermal stresses due to the daily or annual change of the earth temperature are negligibly small.

As for example, if the daily change of the earth temperature at the surface is given by

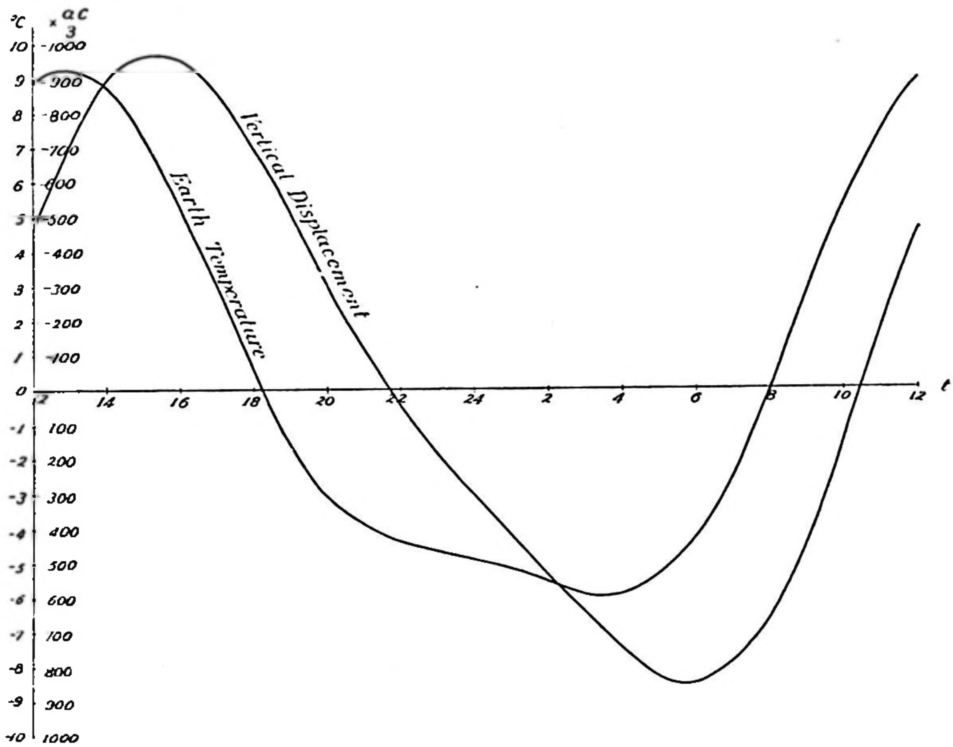

Fig. I.

$$T_{z=0}=7{,}^{\circ}33\sin(\omega t+71^{\circ}30')+2{,}^{\circ}15\sin(2\omega t+71^{\circ}40'),$$ [1]

then the corresponding displacement becomes

$$v=\frac{ac}{3}\left\{\frac{7{,}^{\circ}33}{\sqrt{\omega}}e^{-\frac{x}{a}\sqrt{\frac{\omega}{2}}}\sin\left(\omega t-\frac{x}{a}\sqrt{\frac{\omega}{2}}+206^{\circ}30'\right)\right.$$
$$\left.+\frac{2{,}^{\circ}15}{\sqrt{2\omega}}e^{-\frac{x}{a}\sqrt{\omega}}\sin\left(2\omega t-\frac{x}{a}\sqrt{\omega}+206^{\circ}40'\right)\right\}.$$

For this case, the temperature and the displacement at the surface are tabulated in Table I and illustrated in Fig. I.

Table I.

Time	12^h	14^h	16^h	18^h	20^h	22^h	24^h	2^h	4^h	6^h	8^h	10^h
Earth Temperature (°C)	8.99	8.79	5.06	0.29	−3.07	−4.42	−4.91	−5.58	−5.92	−4.37	−0.14	5.29
Displacement $\left(\times\frac{ac}{3}\right)$	−464	−895	−956	−689	−296	46	304	539	760	849	652	150

§3. For the present purpose, the solution for a three dimensional case in a semi-infinite elastic body obtained in my first paper may be modified as follows:

To the particular solutions in the 1st paper

$$\left.\begin{aligned}u_0&=Pe^{-a^2(\alpha^2+\beta^2+\gamma^2)t}\sin\alpha(x-\xi)\cos\beta(y-\eta)\sin\gamma z\sin\gamma\zeta\\ v_0&=Qe^{-a^2(\alpha^2+\beta^2+\gamma^2)t}\cos\alpha(x-\xi)\sin\beta(y-\eta)\sin\gamma z\sin\gamma\zeta\\ w_0&=Re^{-a^2(\alpha^2+\beta^2+\gamma^2)t}\cos\alpha(x-\xi)\cos\beta(y-\eta)\cos\gamma z\sin\gamma\zeta\end{aligned}\right\},$$

we must add the components of displacement satisfying

$$(\lambda+2\mu)\operatorname{grad}\operatorname{div}\mathfrak{s}-\mu\operatorname{rot}\operatorname{rot}\mathfrak{s}=\rho\frac{\partial^2\mathfrak{s}}{\partial t^2}$$

as follows:

$$\left.\begin{aligned}u_1&=P_1e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}e^{-\gamma_1z}\sin\alpha(x-\xi)\cos\beta(y-\eta)\\ v_1&=Q_1e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}e^{-\gamma_1z}\cos\alpha(x-\xi)\sin\beta(y-\eta)\\ w_1&=R_1e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}e^{-\gamma_1z}\cos\alpha(x-\xi)\cos\beta(y-\eta)\end{aligned}\right\},$$

and

$$\left.\begin{aligned}u_2&=P_2e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}e^{-\gamma_2z}\sin\alpha(x-\xi)\cos\beta(y-\eta)\\ v_2&=Q_2e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}e^{-\gamma_2z}\cos\alpha(x-\xi)\sin\beta(y-\eta)\\ w_2&=R_2e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}e^{-\gamma_2z}\cos\alpha(x-\xi)\cos\beta(y-\eta)\end{aligned}\right\},$$

(1) This expression was used by R. Kuraisi in his recent paper "On the Conduction of Heat in a Semi-infinite Solid partially covered with Insulating Material." Geophys. Mag., Okada Volume, p. 344 (1932).

where

$$P_1 : \alpha = Q_1 : \beta = R_1 : \gamma_1 , \tag{3.1}$$

$$P_2\alpha + Q_2\beta - R_2\gamma_2 = 0, \tag{3.2}$$

and

$$\gamma_1^2 = \frac{a^4(\alpha^2+\beta^2+\gamma^2)^2}{\lambda+2\mu}\rho + \alpha^2+\beta^2, \tag{3.3}$$

$$\gamma_2^2 = \frac{a^4(\alpha^2+\beta^2+\gamma^2)^2}{\mu}\rho + \alpha^2+\beta^2. \tag{3.4}$$

The first set is the expressions for the displacement satisfying rot $\mathfrak{s}=0$, while the second set is the expressions for the displacement satisfying div $\mathfrak{s}=0$. From the conditions $Z_z=Z_x=Z_y=0$ at $z=0$, we have

$$\left.\begin{aligned} &\lambda(P_1\alpha+Q_1\beta-R_1\gamma_1)-2\mu R_1\gamma_1-2\mu R_2\gamma_2=0 \\ &-P_1\gamma_1-R_1\alpha-P_2\gamma_2-R_2\alpha+(P\gamma-R\alpha)\sin\gamma\zeta=0 \\ &-Q_1\gamma_1-R_1\beta-Q_2\gamma_2-R_2\beta+(Q\gamma-R\beta)\sin\gamma\zeta=0 \end{aligned}\right\} . \tag{3.5}$$

By virtue of (3.1), (3.2) and (3.5), P_1, Q_1, R_1 ; P_2, Q_2 and R_2 can be expressed in terms of P, Q and R. Thus if we start from any given distribution of invariable heat we can obtain the required solutions, by adding u_1 and u_2 to u_0 etc. and integrating as follows:

$$u = \frac{2}{\pi^3}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\int_0^{\infty}d\zeta\int_0^{\infty}d\alpha\int_0^{\infty}d\beta\int_0^{\infty}d\gamma. \tag{3.6}$$

$$\{P[f(\xi,\eta,\zeta)]\sin\gamma z\sin\gamma\zeta + P_1e^{-\gamma_1 z}+P_2e^{-\gamma_2 z}\}e^{-a^2(\alpha^2+\beta^2+\gamma^2)t}\sin\alpha(x-\xi)\cos\beta(y-\eta), \text{ etc.}$$

As shown above, the effect due to the time variation of heat is very small and may be neglected. Hence we can safely discuss the effect of temperature by calculating the deformation due to the distribution of temperature at the instant under consideration. If the distribution of heat in a semi-infinite elastic body is given by any given function $f(x, y, z)$, then the displacements are given by, by putting $\rho=t=0$, in Eqs. (3.3), (3.4), (3.6),

$$u = \frac{2}{\pi^3}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\zeta\int_0^{\infty}d\alpha\int_0^{\infty}d\beta\int_0^{\infty}d\gamma$$

$$\{P[f(\xi,\eta,\zeta)]\sin\gamma z\sin\gamma\zeta + P_1e^{-\gamma_1 z}+P_2e^{-\gamma_2 z}\}\sin\alpha(x-\xi)\cos\beta(y-\eta), \text{ etc.,}$$

where

$$P[f(\xi,\eta,\zeta)] = \frac{\left(\lambda+\frac{2}{3}\mu\right)cf(\xi,\eta,\zeta)}{\Delta}\begin{vmatrix} \alpha & (\lambda+\mu)\alpha\beta & -(\lambda+\mu)\alpha\gamma \\ \beta & (\lambda+2\mu)\beta^2+\mu(\alpha^2+\gamma^2) & -(\lambda+\mu)\beta\gamma \\ -\gamma & -(\lambda+\mu)\beta\gamma & (\lambda+2\mu)\gamma^2+\mu(\alpha^2+\beta^2) \end{vmatrix},$$

$$\Delta=\begin{vmatrix} (\lambda+2\mu)\alpha^2+\mu(\beta^2+\gamma^2) & (\lambda+\mu)\alpha\beta & -(\lambda+\mu)\alpha\gamma \\ (\lambda+\mu)\alpha\beta & (\lambda+2\mu)\beta^2+\mu(\alpha^2+\gamma^2) & -(\lambda+\mu)\beta\gamma \\ -(\lambda+\mu)\alpha\gamma & -(\lambda+\mu)\beta\gamma & (\lambda+2\mu)\gamma^2+\mu(\alpha^2+\beta^2) \end{vmatrix},$$

$$\gamma_1^2=\gamma_2^2=\alpha^2+\beta^2.$$

Hence the solution for the most general case is obtained. The solution for two dimensional case can similarly be obtained.